The Exact Solution of one Fokker-Planck Type Equation used by R.Friedrich and J.Peinke in the Stohastic Model of a Turbulent Cascade

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Abstract

The exact solution of the Cauchy problem for a Fokker-Planck equation used by R.Friedrich and J.Peinke for the description of a turbulent cascade, considered as a stochastic process of Markovian type, is obtained in the frame of M.Suzuki approach.

1 Introduction

The understanding of the turbulence is one of the main unsolved problems of classical physics, in spite of the more than 250 years of strong investigations initiated by D.Bernoulli and L.Euler.

In the stochastic approach to turbulence [1], [2] the turbulent cascade is considered as a stochastic process, described by the probability distribution $P(\lambda, v)$, where λ and v are the appropriate scaled length and velocity increment respectively. Recently [3] R.Friedrich and J.Peinke presented experimental evidence that the probability density function $P(\lambda, v)$ obeys a Fokker-Planck equation (FPE) [4] (see fig.1 and fig.2 in [3]):

$$\frac{\partial P(\lambda,v)}{\partial \lambda} = \left[-\frac{\partial}{\partial v} D^1(\lambda,v) + \frac{\partial^2}{\partial v^2} D^2(\lambda,v)\right] P(\lambda,v), \tag{1}$$

where the drift and duffusion coefficients D^1 and D^2 respectively are derived by analysis of experimental data of a fluid dynamical experiment (see fig.3 in [3]).

In their paper Friedrich and Peinke consider the application of the FPE to obtain the Kolmogorov scaling with simplified assumptions that D^1 and D^2 are λ-independent, D^1 is linear in v and D^2 is quadratic in v:

$$D^1 = -a\, v, \qquad a > 0; \qquad D^2 = c\, v^2, \qquad c > 0\,.$$

(In the notations of [3] : $a \equiv \gamma$ and $c \equiv Q$.)

Here we will consider a more realistic situation (see fig.3 in [3]) of λ-dependent D^1 and D^2 :

$$D^1 = -a(\lambda)\, v, \qquad a(\lambda) > 0; \qquad D^2 = c(\lambda)\, v^2, \qquad c(\lambda) > 0. \tag{2}$$

Thus the FPE (1) will take the form:

$$\frac{\partial P}{\partial \lambda} = b_0(\lambda) P(\lambda, v) + b_1(\lambda) v \frac{\partial P}{\partial v} + c(\lambda) \left(v \frac{\partial}{\partial v} \right)^2 P(\lambda, v), \tag{3}$$

where

$$b_0(\lambda) = a(\lambda) + 2c(\lambda), \quad b_1(\lambda) = a(\lambda) + 3c(\lambda). \tag{4}$$

2 Exact Solution of the Cauchy Problem for the Eq. (3)

In this section we will find the solution $P(\lambda, v)$ of the Cauchy problem for the Eq. (3) with the initial condition

$$P(0, v) = \varphi(v). \tag{5}$$

According to [1]$-$[3], when the probability density function is known, one may derive all properties of the turbulent cascade considered as a stochastic process.

For the solution of the problem (3), (5) we shall use the approach of M.Suzuki [5] to the FPE (see also [6]), based on the disentangling techniques of R.Feynman [7] and the operational methods developed in the functional analysis, in particular in the theory of pseudodifferential equations with partial derivatives [8]$-$[12]

In the spirit of the operational methods using the pseudodifferential operators we can write the solution of the Cauchy problem (3), (5) in the form

$$P(\lambda, v) = \left(exp_+ \int_0^\lambda \left[b_0(s) + b_1(s) v \frac{\partial}{\partial v} + c(s) \left(v \frac{\partial}{\partial v} \right)^2 \right] \mathrm{d}s \right) \varphi(v), \tag{6}$$

where the symbol exp_+ designates the V.Volterra ordered exponential

$$exp_+ \int_0^\lambda \hat{C}(s) \mathrm{d}s = \hat{1} + \lim_{n \to \infty} \sum_{k=1}^{n} \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \ldots \int_0^{\lambda_{k-1}} \mathrm{d}\lambda_k \hat{C}(\lambda_1) \hat{C}(\lambda_2) \ldots \hat{C}(\lambda_k). \tag{7}$$

The linearity of the integral and the explicit form of the operators in (6) permit to write the solution $P(\lambda, v)$ in terms of usual, not ordered, operator valued exponent

$$P(\lambda, v) = \mathrm{e}^{\beta_0(\lambda)}\, \mathrm{e}^{\beta_1(\lambda) v \frac{\partial}{\partial v} + \gamma(\lambda) \left(v \frac{\partial}{\partial v} \right)^2} \varphi(v), \tag{8}$$

where for convenience we have denoted

$$\beta_j(\lambda) = \int_0^\lambda b_j(s)\mathrm{d}s, \quad (j=0,1); \qquad \gamma(\lambda) = \int_0^\lambda c(s)\mathrm{d}s. \tag{9}$$

Consequently (from now on "$'$" means $\frac{\mathrm{d}}{\mathrm{d}t}$) :

$$\beta_j(0) = 0, \quad \beta_j'(\lambda) = b_j(\lambda), \quad (j=0,1); \qquad \gamma(0) = 0, \quad \gamma'(\lambda) = c(\lambda). \tag{10}$$

Since the operators $\hat{A} \equiv \beta_1(\lambda) v \frac{\partial}{\partial v}$ and $\hat{B} \equiv \gamma(\lambda)\left(v\frac{\partial}{\partial v}\right)^2$ commute : $[\hat{A},\, \hat{B}] = 0$, from Eq. (8) we have

$$P(\lambda, v) = \mathrm{e}^{\beta_0(\lambda)}\, \mathrm{e}^{\beta_1(\lambda) v \frac{\partial}{\partial v}}\, \mathrm{e}^{\gamma(\lambda)\left(v\frac{\partial}{\partial v}\right)^2} \varphi(v). \tag{11}$$

Therefore, taking into account the formulae

$$\mathrm{e}^{\beta_1(\lambda) v \frac{\partial}{\partial v}} f(v) = f\left(v\mathrm{e}^{\beta_1(\lambda)}\right) \tag{12}$$

and

$$\begin{aligned} \mathrm{e}^{\gamma(\lambda)\left(v\frac{\partial}{\partial v}\right)^2} g(v) &= \frac{1}{\sqrt{4\pi\gamma(\lambda)}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{s^2}{4\gamma(\lambda)}} g\left(v\mathrm{e}^{-s}\right) \mathrm{d}s \\ &= \frac{1}{\sqrt{4\pi\gamma(\lambda)}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(\ln v - y)^2}{4\gamma(\lambda)}} g\left(\mathrm{e}^y\right) \mathrm{d}y, \end{aligned} \tag{13}$$

we obtain the following expression for the exact solution of the Cauchy problem (3),(5)

$$\begin{aligned} P(\lambda, v) &= \frac{\mathrm{e}^{\beta_0(\lambda)}}{\sqrt{4\pi\gamma(\lambda)}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{s^2}{4\gamma(\lambda)}} \varphi\left(v\mathrm{e}^{\beta_1(\lambda) - s}\right) \mathrm{d}s \\ &= \frac{\mathrm{e}^{\beta_0(\lambda)}}{\sqrt{4\pi\gamma(\lambda)}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(\ln v + \beta_1(\lambda) - y)^2}{4\gamma(\lambda)}} g\left(\mathrm{e}^y\right) \mathrm{d}y, \end{aligned} \tag{14}$$

where $\beta_0(\lambda), \beta_1(\lambda)$ and $\gamma(\lambda)$ are defined in (9).

Substituting the expression (14) in the Eqs. (3) and (5) and using the Eq. (10) one can see immediately that $P(\lambda, v)$ is a solution of the problem (3), (5) and, according to the Cauchy theorem, it is the only classical solution of this problem.

3 Concluding remarks

- The exact solution of the Cauchy problem (3), (5) is obtained using the algebraic method we have described. The Eq. (3) is a generalization of the equation used by R.Friedrich and J.Peinke (see section 1) in their description of a turbulent cascade by a Fokker-Planck equation with coefficients derived by a detailed analysis of experimental data of a fluid dynamical experiment.

- If the probability distribution function $P(\lambda, v)$ is known, then one may derive the properties of a given stochastic process, in our case - the turbulent cascade $[1] - [3]$.
- For more realistic description of the turbulent cascade by a FPE it should be desirable to use for $D^1(\lambda, v)$ and $D^2(\lambda, v)$ in the Eq. (1) more general expressions than these in Eq. (2), for instance:
$D^1(\lambda, v) = a_1(\lambda) - a(\lambda)v, \; a(\lambda) > 0$ and $D^2(\lambda, v) = c_1(\lambda) + c(\lambda)v^2$.

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